Answer the following questions:

Question (1)

(a) Test the series $\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$ for convergence and find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n.2^n}$

(b) Given $w = \tan^{-1}(x^3 + y^3)$ Show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 3\sin w \cos w$

(c) Find the local extrema of the function $f(x,y) = -x^2 - 4x - y^2 + 2y - 1$.

Answer (a)

$\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$ we have $a_n = \frac{3n+1}{2^n}$ $\therefore a_{n+1} = \frac{3n+4}{2^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} [\frac{3n+4}{2^{n+1}} \div \frac{3n+1}{2^n}] = \lim_{n\to\infty} \frac{(3n+4)\,2^n}{2^{n+1}(3n+1)}$$

$$= \lim_{n\to\infty} \frac{(3n+4)}{2\ (3n+1)} = \frac{1}{2} < 1$$ Then the series is convergent series.

To determine the interval of convergence for $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n.2^n}$ we apply the ratio test **where** $|u_n| = \frac{(x+3)^n}{n.2^n}$, $|u_{n+1}| = \frac{(x+3)^{n+1}}{(n+1).2^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(x+3)^{n+1}}{(n+1).2^{n+1}} . \frac{n.2^n}{(x+3)^n}\right| = \lim_{n\to\infty} \frac{n}{(n+1).2}|(x+3)| = \frac{|(x+3)|}{2}$$

For convergence put $\frac{|(x+3)|}{2} < 1$ Then the interval of convergence $-5 < x < -1$

Answer (b)

$w = \tan^{-1}(x^3 + y^3)$ Then $\tan w = (x^3 + y^3)$

Differentiate w.r.to x

$$\sec^2 w\ \frac{\partial w}{\partial x} = 3x^2 \rightarrow \frac{\partial w}{\partial x} = 3x^2 \cos^2 w \rightarrow x\ \frac{\partial w}{\partial x} = 3x^3 \cos^2 w$$

$$\sec^2 w\ \frac{\partial w}{\partial y} = 3y^2 \rightarrow \frac{\partial w}{\partial y} = 3y^2 \cos^2 w \rightarrow y\ \frac{\partial w}{\partial y} = 3y^3 \cos^2 w$$

$$x\ \frac{\partial w}{\partial x} + y\ \frac{\partial w}{\partial y} = 3x^3 \cos^2 w + 3y^3 \cos^2 w = 3\left(x^3 + y^3\right)\cos^2 w$$

$$= 3\tan w\ \cos^2 w = 3\sin w\ \cos w$$

Answer (c)

$$f(x,y) = -x^2 - 4x - y^2 + 2y - 1$$

$$\frac{\partial f}{\partial x} = -2x - 4, \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y + 2,$$

Since f_x and f_y exist for every (x, y) the only critical points are the solution of the following system of two equation in two variables

$$\frac{\partial f}{\partial x} = -2x - 4 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y + 2 = 0$$

which is the point $(-2,1)$

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial y\, \partial x} = 0$$

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial y\, \partial x}\right]^2 = (-2)(-2) - (0)^2 = 4 > 0$$

Since $\frac{\partial^2 f}{\partial x^2} = -2 < 0$ then $f(-2,1)$ is a local maximum for the function $f(x,y)$

Question (2)

(a) For any scalar function $\varphi(x, y, z)$ show that $curl\ grad\varphi = 0$

(b) Find the area enclosed by the following curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(c) Find the are bounded by the curves $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$

Answer (a)

$$\text{curl grad}\ \phi = \vec{\nabla}\times(\vec{\nabla}\phi) = \vec{\nabla}\times\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\vec{k}$$

$$=\left[\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right]\vec{i}+\left[\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right]\vec{j}+\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right]\vec{k}=0$$

Answer (b)

The parametric equation is $\boldsymbol{x = a\cos^3 t,\ \ y = a\sin^3 t}$

$$x = a\cos^3 t \quad\Rightarrow\quad dx = -3a\cos^2 t\sin t\,dt$$
$$y = a\sin^3 t \quad\Rightarrow\quad dy = 3a\sin^2 t\cos t\,dt$$

$$\iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$$

$$=\frac{1}{2}\int_0^{2\pi}\left[(a\cos^3 t)(3a\sin^2 t\cos t)\right]dt-\left[(a\sin^3 t)(-3a\cos^2 t\sin t)\right]dt$$

$$=\frac{3a^2}{2}\int_0^{2\pi}\left[(\cos^3 t)(\sin^2 t\cos t)\right]dt+\left[(\sin^3 t)(\cos^2 t\sin t)\right]dt$$

$$=\frac{3a^2}{2}\int_0^{2\pi}\left[\sin^2 t\cos^4 t+\cos^2 t\sin^4 t\right]dt=\frac{3a^2}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\,dt$$

$$=\frac{3a^2}{8}\int_0^{2\pi}\sin^2 2t\,dt=\frac{3a^2}{16}\int_0^{2\pi}(1-\cos 4t)dt=\frac{3a^2}{16}2\pi=\boxed{\frac{3a^2\pi}{8}}$$

Answer (c)

Put $u = xy = 4,\quad v = xy^3$ then $dxdy = J\left(\frac{u,v}{x,y}\right)dudv = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dudv$

$$=\begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} dudv = \left(3xy^3 - xy^3\right) dudv = 2v dudv$$

The area bounded by the curves given by

$$\iint_A dxdy = \int_5^{15} \int_4^8 2v dudv = v^2 \Big|_5^{15} u \Big|_4^8 = \left(15^2 - 5^2\right)(8-4) = \boxed{400 \text{ area unit}}$$

Question (3)

Solve the following differential equations:

(a) $(xy - x^2)dy - y^2 dx = 0$

(b) $(xy^3 - 1)dx - x^2y^2 dy = 0$

(c) $y'' - 6y' + 13y = 8e^{3x} \sin 2x$

Answer (a)

$\boldsymbol{M(x,y), N(x,y)}$ are homogeneous of the same degree(second degree)

let $\boldsymbol{y = ux} \quad \therefore \boldsymbol{dy = udx + xdu}$

Substitute in the differential equation we have

$$(x^2u - x^2)(udx + xdu) - x^2u^2dx = 0$$

$$x^2(u-1)(udx + xdu) - x^2u^2dx = 0$$

Divided by $\boldsymbol{x^2}$ we have $\boldsymbol{(u-1)(udx + xdu) - u^2dx = 0}$

$$(u-1)udx + (u-1)xdu - u^2dx = 0$$

$$\left[(u-1)u - u^2\right]dx + (u-1)xdu = 0$$

$-udx + (u-1)xdu = 0$ separate the variables $-\dfrac{dx}{x} + \dfrac{(u-1)}{u}du = 0$

$-\dfrac{dx}{x} + (1 - \dfrac{1}{u})du = 0$ by integration we have $-\ln x + u - \ln u + \ln C = 0$

$\ln \dfrac{xu}{C} = u \qquad \Rightarrow \quad \boxed{y = Ce^{y/x}}$

Another solution

$$(xy - x^2)dy - y^2dx = 0$$

$$xydy - x^2dy - y^2dx = 0$$

$$\left(xydy - y^2dx\right) - x^2dy = 0$$

$$y\left(xdy - ydx\right) - x^2dy = 0$$

$$yx^2d\left(\frac{y}{x}\right) - x^2dy = 0$$ divide by yx^2

$$d\left(\frac{y}{x}\right) - \frac{dy}{y} = 0$$ integrate $\frac{y}{x} - \ln y = \ln C \qquad \frac{y}{x} = \ln Cy \Rightarrow \boxed{y = Ae^{y/x}}$

Answer (b)

$$M = (xy^3 - 1), \qquad N = x^2y^2$$

$$\frac{\partial M}{\partial y} = 3xy^2, \qquad \frac{\partial N}{\partial x} = 2xy^2$$

now $M_x \neq N_x$ Then equation is nonexact. we find the integrating factor

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2 - 2xy^2 = xy^2$

$\frac{1}{N}\left(M_y - N_x\right) = \frac{1}{x}$ as a function of **x**

$$\int \frac{1}{N}\left(M_y - N_x\right)dx = \int \frac{1}{x}dx = \ln x$$

since $\mu(x) = e^{\int \frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$ then $\mu(y) = e^{\ln x} = x$

Multiply the equation by x we have

$x(xy^3 - 1)dx + x^3y^2dy = 0$ which is exact equation and solve it as an exact equation

$$\int_0^x x(xy^3 - 1)dx = c$$

$$\frac{1}{3}x^3y^3 - \frac{1}{2}x^2 = c \qquad \boxed{2x^3y^3 - 3x^2 = C}$$

Another solution

$$(xy^3 - 1)dx + x^2y^2dy = 0$$

$$\left(xy^3dx + x^2y^2dy\right) - dx = 0$$

$$xy^2\left(ydx + xdy\right) - dx = 0$$ multiply by x

$$x^2y^2d\left(xy\right) - xdx = 0$$

By integration $\int (xy)^2 d(xy) - \int x dx = 0$

$$\tfrac{1}{3}(xy)^3 - \tfrac{1}{2}x^2 = c \qquad \Rightarrow \qquad \boxed{2x^3y^3 - 3x^2 = C}$$

Answer (c)

Characteristic equation in the form $m^2 - 6m + 13 = 0$ which has a roots

$$m = \frac{6 \pm \sqrt{36-(4)(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4\sqrt{-1}}{2} = 3 \pm 2i$$

then the general solution in the form $y = e^{3x}(A\cos 2x + B\sin 2x)$

$$y_p = \frac{1}{D^2 - 6D + 13} 8e^{3x} \sin 2x = 8e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 2x$$

$$= 8e^{3x}\left(\frac{1}{(D^2+4)} \sin 2x\right) = 8e^{3x}\left(\frac{-x\cos 2x}{4}\right) = -2xe^{3x}\cos 2x$$

The general solution is $\boxed{y_G = e^{3x}(A\cos 2x + B\sin 2x) - 2xe^{3x}\cos 2x}$

Where A and B are arbitrary constants

Question (4)

(a) Find the general solution for Euler equation $x^2y'' - xy' + 2y = x\ln x$

(b) Use variation of parameter to solve $y'' + n^2y = \sec nx.$

(c) Solve $xy'' - (2x+1)y' + (x+1)y = 0$ given that $y = e^x$ is a one solution

Answer (a)

(a) Put $x = e^t$ in the given equation then $t = \ln x$ and use the fact that

$xy' = \dfrac{dy}{dt} = Dy$ and $x^2y'' = D(D-1)y$ then equation (4) transform to

$$(D^2 - 2D + 2)y = te^t \qquad (5)$$

which is linear differential equation with constant coefficient with characteristic equation $(m^2-2m+2)=0$ with roots $m_1,m_2=1\pm i$

$\therefore y_c=e^t(A\cos t+B\sin t)$ where A,B are arbitrary constants

$$y_p=\frac{1}{D^2-2D+2}te^t=e^t\frac{1}{(D+1)^2-2(D+1)+2}t$$

$$=e^t\frac{1}{(D^2+1)}t=e^t(1+D^2)^{-1}t=e^t(1-D^2+O(D^4))\;t=te^t$$

$$\therefore y_G=e^t(A\cos t+B\sin t)+te^t \qquad (6)$$

which is the general solution of equation (5) put $t=\ln x$ in (6) then the general solution of equation (4) is

$$\therefore \boxed{y_G=x(A\cos\ln x+B\sin\ln x)+x\ln x}$$

Answer (b)

The characteristic equation for the homogeneous equation in the form $m^2+n^2=0$ which has a roots are $m=\pm ni$ then the solution

$$y_c=A\cos nx+B\sin nx \qquad (2)$$

where A, B are arbitrary constant

let the general solution

$$y_G=A(x)\cos nx+B(x)\sin nx \qquad (3)$$

Subject to $\quad A'\cos nx+B'\sin nx=0 \qquad (4)$

Then $y'=-nA\sin nx+nB\cos nx \qquad (5)$

$$y''=-nA'\sin nx-n^2A\cos nx+nB'\cos nx-n^2B\sin nx$$

Substitute in (1)

$$-nA'\sin nx+nB'\cos nx=\sec nx \qquad (6)$$

Solve (4) and (6)

$$B' = 1/n \qquad \to \qquad B(x) = x/2 + C_1$$

$$A' = -\frac{1}{n}\tan nx \;\to\; A = \frac{1}{n^2}\ln\cos nx + C_2$$

Substitute in (3)

$$y_G = \left(\frac{1}{n^2}\ln\cos nx + C_2\right)\cos nx + \left(\frac{x}{2} + C_1\right)\sin nx$$

$$= \left(\frac{\cos nx}{n^2}\ln\cos nx + C_2\cos nx\right) + \left(\frac{x}{2}\sin nx + C_1\sin nx\right)$$

$$\boxed{y_G = C_1\sin nx + C_2\cos nx + \frac{\cos nx}{n^2}\ln\cos nx + \frac{x}{2}\sin nx}$$

Answer (c)

Let the general solution $y = v\,e^x$

$$y' = v'e^x + ve^x$$

$$y'' = v''e^x + 2v'e^x + ve^x$$

Substitute in the homogeneous equation

$$xy'' - (2x+1)y' + (x+1)y = 0$$

$$v''xe^x + 2v'xe^x + vxe^x - (2x+1)(v'e^x + ve^x) + (x+1)ve^x = 0$$

$$v''x - v' = 0$$

$$\frac{v''}{v'} = \frac{1}{x} \Rightarrow v' = c_1 x \Rightarrow v = \tfrac{1}{2}c_1x^2 + c_2 \text{ then } \boxed{y_c = \tfrac{1}{2}c_1x^2e^x + c_2e^x}$$

Question (5)

(a) For the vector field $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\,\vec{k}$ Prove that $\oint_C \vec{F}.d\vec{r}$ independent to any path through two any point and Find φ such that $\vec{F} = \vec{\nabla}\varphi$.

(b) Evaluate $\iint_S \vec{F}.\vec{n}\,ds$ where $\vec{F} = 2yx\,\vec{i} + yz^2\,\vec{j} + xz\,\vec{k}$ in the surface of parallelogram bounded by $x=0,\ y=0,\ z=0\ \ x=2,\ y=1,\ z=3$

(c) Apply Stock and Green theorem to evaluate $\iint_S (\overline{\nabla} \times \overline{F}).\overline{n}\, dS$ where

$\vec{F} = (x^2 + y - 4)\vec{i} + (3xy)\vec{j} + (2xz + z^2)\vec{k}$ and $\boldsymbol{S}$ is the surface bounded by the

paraboloid $z = 4 - (x^2 + y^2),\ x \geq 0$.

Answer (a)

The line integral $\int_C \vec{F}.d\vec{r}$ independent to any path through two any point in

domain $\vec{F}$ If $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4yx - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} = 0$$

Then the field $\vec{F}$ is conservative and $\int_C \vec{F}.d\vec{r}$ independent to any path through two point in domain $\vec{F}$ Moreover there exist scalar function φ such that $\vec{F} = \vec{\nabla}\varphi$ thus:

$$F.dr = \nabla\varphi.dr = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi$$

$$\therefore d\varphi = F.dr = (4yx - 3x^2z^2)dx + 2x^2dy - 2x^3z\,dz$$

$$= (-3x^2z^2dx - 2x^3zdz) + (4yxdx + 2x^2dy)$$

$$= d(-x^3z^2) + d(2x^2y) = d(-x^3z^2 + 2x^2y)$$

$$\therefore \boxed{\varphi = -x^3z^2 + 2x^2y + c}$$ c is a constsnt

Answer (b)

$$\therefore \oint\!\!\!\oint_S \vec{F}.\vec{n}ds = \iiint_V \vec{\nabla}.\vec{F}\, dV =$$

$$\vec{F} = 2yx\,\vec{i} + y\,z^2\,\vec{j} + xz\,\vec{k}$$

$$\overline{\nabla}.\overline{F} = 2y + z^2 + x$$

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}\,dV = \iiint_V \left[2y + z^2 + x\right]dxdydz$$

$$= \int_0^3 \int_0^1 \int_0^2 \left[2y + z^2 + x\right]dxdydz = \int_0^3 \int_0^1 \left[2yx + z^2x + \frac{1}{2}x^2\right]_0^2 dydz$$

$$= \int_0^3 \int_0^1 \left[4y + 2z^2 + 2\right]dydz = \int_0^3 \left[2y^2 + 2z^2y + 2y\right]_0^1 dz$$

$$= \int_0^3 \left[4 + 2z^2\right]dz = \left[4z + \frac{2}{3}z^3\right]_0^3 = 12 + 18 = \boxed{30}$$

Answer (c)

First Applying Stock Theorem $\iint_S (\vec{\nabla} \times \vec{F}).\vec{n}\,dS = \oint_C F.dr$

where C is the boundary of the surface (circumference of the circle $x^2 + y^2 = 4$) in $xy - plane$

$$\oint_C F.dr = \int_0^{2\pi} (x^2 + y - 4)dx + (3xy)\,dy + (2xz + z^2)dz$$

use the parametric equation $z = 0$

$$x = 2\cos\theta \Rightarrow dx = -2\sin\theta d\theta \quad and \quad y = 2\sin\theta \Rightarrow dy = 2\cos\theta d\theta$$

$$\oint_C F.dr = \int_0^{2\pi} (x^2 + y - 4)dx + (3xy)\,dy$$

Second apply Green theorem and Use polar coordinates

$x = r\cos\theta,\ y = r\sin\theta$

$$\oint_C F.dr = \int_0^{2\pi} (x^2 + y - 4)dx + (3xy)\,dy = \iint (3y - 1)dxdy$$

$$= \int_0^{2\pi}\int_0^2 (3r\sin\theta - 1)rdrd\theta = \int_0^{2\pi}\int_0^2 3r^2\sin\theta drd\theta - \int_0^{2\pi}\int_0^2 rdrd\theta = \boxed{-4\pi}$$

Dr. Fathi Abdessalam